A Philosopher's[1] Thoughts[2] on Set Theory[3]

Sara L. Uckelman
Durham University
s.l.uckelman@durham.ac.uk
@SaraLUckelman

Set Theory in the UK
Leeds, England
15 May 2026

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3. To be precise, set theory *and sets*, as IANASetTheorist myself. . .

Three Things I Hope You'll Find Interesting

- Set theory without sets
- Set theory without theory
- Set theory with sets and theory!

Infinity and cardinality in the Middle Ages (1)

Adam of Balsham, *Ars disserendi* c1132:

> *It can be asked whether this one is something, and whether that one is something, and these two are askables [that which can be asked]; and also whether a third is something, and similarly of anything whatever. . . But this question can be put not only about those which are not askables; of any askable too it can be asked whether it is something. Whether the questions be actually put or not, makes no difference; even when they are not put, the points remain askable. Thus the totality of askables is seen to be equinumerous with the totality of objects, which last comprises both the objects which are not askables and those which are [quote/translation adapted from [5]].*

Infinity and cardinality in the Middle Ages (2)

A similar puzzle arises with enuntiables [that which can be said/asserted], which are said to be either true or false. For every true enuntiable, there is another enuntiable that says that first one is true, regardless of whether any of these enuntiables are ever stated.

Infinity and cardinality in the Middle Ages (3)

Adam thinks both of these puzzles are impossible (*quod est impossibile*), and points out they cannot arise when only finite constructions are considered, and not those which infinitely exceed our capability to construct:

> *hoc interim compertori plene sufficiente quod nos artis viam finitam, non possibilitas excessum infinitum cognoscibilem reddere proposuimus [5, p. 134].*

(cf. Aristotelian potential vs. actual infinities).

The paradox of unequal infinites (1)

Let AB be a line beginning at some arbitrary point A and extended ad infinitum toward point B. Let C be a point on the line a finite distance from A. Then line CB is a proper part of AB; yet both are equally infinitely long.

The paradox of unequal infinites (1)

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Some solutions:

- Albert of Saxony, Nicholas Oresme: ‘equal to,’ ‘greater than,’ and ‘less than’ don’t apply to actual infinites, only to finite objects.
- lines AB and CB are “unequal relative to their termini” — equality is directed.

The paradox of unequal infinites (2)

Some more solutions:

- Henry of Harclay (c1312): there can be unequal infinites, we must figure out what part-whole axiom governs these. (The Euclidean "Every whole is greater than its parts" only applies to finite objects).
- William of Alnwick (d1333): "beyond" entails having more absolutely; "in addition to" entails greater diversity but not necessarily greater plurality.
- Gregory of Rimini (d1358): two ways to understand part/whole relationships; (1) "a whole which includes something that is something and something else in addition to that something"; (2) "a whole includes something in the first thing and also includes as many thing as that included does not include" [3, p. 572]

Sets without theorizing

- Sets as logical tools.
- Sets as metaphysical tools.
- Sets as metaphysical problems.

Sets as logical tools

- Tarskian model theory
  - for predicate logic,
  - for Aristotelian syllogistic,
  - for building models of medieval logic

Sets as metaphysical tools

Take a philosophical/metaphysical notion, express it via sets:

- Possible worlds as sets of sentences
- Predicates as sets of objects

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Sets as metaphysical tools

Take a philosophical/metaphysical notion, express it via sets:

- Possible worlds as sets of sentences —what about impossible worlds?
- Predicates as sets of objects —intension vs. extension

Sets as metaphysical problems

- What even are sets?
- Where are they located?
- How do we know things about them?

Having our cake *and* eating it too

- Modal logic of forcing

Having our cake *and* eating it too

- Modal logic of forcing
  - modal logic
  - forcing
  - how the two combine

Modal logic (1)

- Logic of $\Box$ (necessity) and $\Diamond$ (possibility).
- Non-truth functional.
- Standard semantics: Possible worlds semantics with models $\mathfrak{M} = \langle W, R, V \rangle$.
- Many different ways of interpreting $\Box\varphi$ "it is necessary that φ" and $\Diamond\varphi$ "it is possible that φ."
- Different modal principles correspond to different frame principles (e.g., reflexivity, transitivity, symmetry, connectedness, etc.)

Modal logic (2)

$\mathfrak{M} = \langle \mathfrak{F} = \langle W, R \rangle, V \rangle$ where

- W is a set (of possible worlds).
- R is a binary (accessibility) relation on W.
- V is a valuation assigning each atomic proposition p a truth value (either T or F) at each world $w \in W$.

Modal logic (3)

Truth is defined recursively with respect to worlds:

$\mathfrak{M}, w \vDash p$	iff	$V(p, w) = T$
$\mathfrak{M}, w \vDash \neg\varphi$	iff	$\mathfrak{M}, w \nvDash \varphi$
$\mathfrak{M}, w \vDash \varphi \wedge \psi$	iff	$\mathfrak{M}, w \vDash \varphi$ and $\mathfrak{M}, w \vDash \psi$
$\mathfrak{M}, w \vDash \varphi \vee \psi$	iff	$\mathfrak{M}, w \vDash \varphi$ or $\mathfrak{M}, w \vDash \psi$
$\mathfrak{M}, w \vDash \varphi \rightarrow \psi$	iff	either $\mathfrak{M}, w \nvDash \varphi$ or $\mathfrak{M}, w \vDash \psi$
$\mathfrak{M}, w \vDash \Diamond\varphi$	iff	there is a w' such that $wRw'$and $\mathfrak{M}, w' \vDash \varphi$
$\mathfrak{M}, w \vDash \Box\varphi$	iff	for every w' such that $wRw', \mathfrak{M}, w' \vDash \varphi$

Modal principles and correspondence

- Necessity implies truth — R is reflexive
- Necessity is always itself necessary — R is transitive.
- Possible necessity is necessarily possible — R is convergent.

Forcing

- A method for constructing models of set theory.
- Take a model V of set theory, add a V-generic filter G over a partial order $\mathbb{P}$ in the ground model V, and get new numbers!
- By picking G appropriately, it is possible to ensure that $V[G]$ (the "forcing extension of V with G") has appropriate/desired properties.

Modal logic of forcing

- Introduced by Hamkins and Löwe [2].
- Interpret $\Diamond\varphi$ as "there is a forcing extension in which φ is true" and $\Box\varphi$ as "in every forcing extension, φ is true".
- What types of forcing correspond to which modal logics?
- "Every definable forcing class Γ gives rise to a corresponding forcing modality. . . and the valid principles of Γ forcing are the modal assertions that are valid for this forcing interpretation" [1, p. 618].

Some results (1)

- Hamkins and Löwe [2]: the modal logic of the ZFC-provably valid principles for the class of all forcing is S4.2. S4.2 is the logic containing the following axioms:
  - $\mathrm{K} := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
  - $\mathrm{Dual} := \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$.
  - $\mathrm{T} := \Box\varphi \to \varphi$ (reflexivity)
  - $4 := \Box\varphi \to \Box\Box\varphi$ (transitivity)
  - $.2 := \Diamond\Box\varphi \to \Box\Diamond\varphi$ (convergence)

Some results (2)

- Rittberg [4]: the modal logic of ω-closed forcing is also S4.2, and so is $\leq_\kappa$-closed forcing where κ is definable by φ.
- Hamkins, Leibman, Löwe [1]: the modal logic of countably closed forcing, CH-preserving forcing, (and some others) is also S4.2.
- HLL [1]: the modal logic of collapse forcing, Cohen forcing, (and some others) is S4.3, where S4.3 is the same as S4.2 except that axiom $.2$ is swapped for $.3$:

$$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond((\varphi \wedge \Diamond\psi) \vee (\psi \wedge \Diamond\varphi)) \text{ (linearity)}$$

Some results (3)

- HLL [1]: the modal logic of c.c.c. forcing, proper forcing, (and some others) is contained in S4.3 and not contained in S4.2.
- HLL [1]: the modal logic of ω_1-preserving forcing is contained within S4.tBA, which is $\mathrm{S4}(:= \mathrm{K}, \mathrm{T}, 4)$ plus "all modal assertions that are true in all Kripke models whose frame is a finite topless pre-Boolean algebra" (p. 622). (A pre-Boolean algebra is a partial pre-order such that the quotient by the relation $x \equiv y \leftrightarrow x \leq y \leq x$ is a Boolean algebra).

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