‘Proof Theory’ in Medieval Logic

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Proofs, Rules, and Meanings
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Semantics vs. syntax

- Setting aside proof-theoretic semantics, modern proof theory is on the “syntax” side of things: an exercise in symbolic manipulation.
- Medieval logic: no notion of symbolic representation = no possibility of symbolic maninpulation!

Motivating question

Q: "What are the medieval antecedents of modern proof theory?"

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A: "there aren't any."

But that's boring

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What is "proof theory" when you don't have a syntax / semantics distinction?

Rules for inferences/consequences that are based on the (grammatical!) syntax of the sentence, not its semantics.

(Brief!) introduction to Latin syntax (1)

In English, order matters:

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$$\text{Margerit loves Barbara} \tag{1}$$

and

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In English, order matters:

$$\text{Margerit loves Barbara} \tag{1}$$

and

$$\text{Barbara loves Margerit} \tag{2}$$

do not have the same meaning, because the order of the names determines which is the subject and which is the object.

(Brief!) introduction to Latin syntax (2)

In Latin, both

$$\text{Margerita amat Barbaram} \tag{3}$$

and

$$\text{Barbaram amat Margerita} \tag{4}$$

mean the same thing,

(Brief!) introduction to Latin syntax (2)

In Latin, both

$$\text{Margerita amat Barbaram} \tag{3}$$

and

$$\text{Barbaram amat Margerita} \tag{4}$$

mean the same thing, and don't mean the same as

$$\text{Margeritam amat Barbara} \tag{5}$$

Order still matters! (1)

Prepositions:

Ad astra per aspera (6)

and

Per astra ad aspera (7)

don't mean the same thing.

Order still matters! (1)

Negation:

Non quaedam feles est varia (8)

Quaedam feles non est varia (9)

Quaedam feles est non varia (10)

Usus loquendi vs. usus proprie

$$\text{All that glitters is not gold} \tag{11}$$

vs.

$$\text{Not all that glitters is gold} \tag{12}$$

Walter Burley

- Born around 1275, studied and taught at Oxford in the early 14th century, died soon after 1344.
- *De Puritate Artis Logicae* (On the Purity of the Art of Logic), published in two versions: an "earlier, shorter version was written before the appearance of Ockham's *Summa Logicae* in 1324" and the second version "dating from 1325–28, is in many ways a response to Ockham's logic" [1, p. 283].
- First two sections of the shorter version on the topic of *consequentiae*, or consequences, "what follows from what".
- Burley was one of the earliest logicians to separate this notion out from other discussions.
- Thus, his text can be taken to be one of the first, if not *the* first, texts on "proof theory," that is, general rules for logical consequence.

Walter Burley.
Consequences (selections).
In Norman Kretzmann and Eleonore. Stump, editors, *Cambridge Translations of Medieval Philosophical Texts*, volume 1: logic and the philosophy of language, pages 283–311. Cambridge University Press, 1988.

Burley's rules for propositional consequences (1)

Rule (2)

Whatever follows from the consequent follows from the antecedent.

Rule (2′)

Whatever is antecedent to the antecedent is antecedent to the consequent.

These two rules are often called, collectively, reasoning "from the first to the last."

Note: ambiguity of 'consequent' and 'antecedent'.

Burley's rules for propositional consequences (2)

A sequent representation:

$$\frac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \Gamma}{\phi \Rightarrow \Gamma}$$

$$\frac{\Delta \Rightarrow \phi \qquad \phi \Rightarrow \psi}{\Delta \Rightarrow \psi}$$

Corollaries of Burley's rules (1)

Corollary (2a)

In any good consequence one can descend under the antecedent to anything that is antecedent to it in respect of the same consequent [1, p. 288].

This rule itself follows from the following principle:

Rule

In a universal proposition, one can [logically] descend under the subject to any suppositum of the subject in respect to the predicate.

(Whether something is a universal proposition is itself a *syntactic* matter.)

Two examples

Example (of Cor 2a)

If a man is running, an animal is running; therefore if Socrates is running, an animal is running [1, p. 288].

Example (of Cor 2a)

If every man is running, Socrates is running; therefore if every animal is running, Socrates is running [1, p. 288].

Corollaries of Burley's rules (2)

Corollary (2b)

In a conditional whose antecedent is a universal proposition, the subject of the antecedent has immobile supposition in respect of the consequent in such a way that one cannot descend under the subject of the antecedent in respect of the consequent; but in a conditional whose antecedent is an indefinite or partial proposition, the subject has confused, distributive, mobile supposition in respect of the consequent [1, p. 288].

Example

(First half of Rule 2b) This does not follow: "If every man is running, Socrates is running; therefore, if Plato is running, Socrates is running" [1, p. 288].
(Second half of Rule 2b) "If a man is running, an animal is running; therefore, if Socrates is running, an animal is running" follows correctly [1, pp. 288–289].

Supposition, really quickly

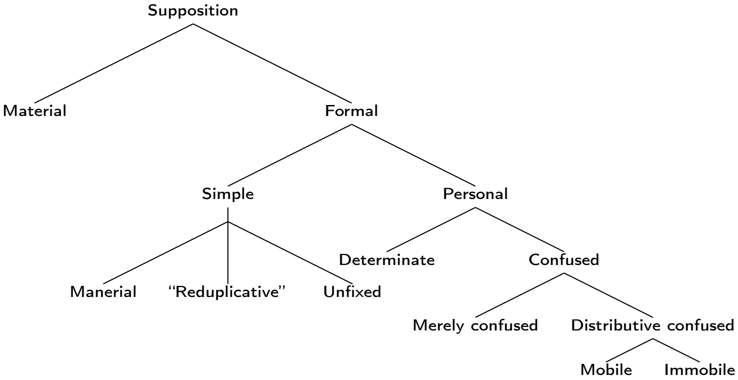

Figure: Sherwood's division of types of *suppositio*.

Corollaries of Burley's rules (3)

Corollary (2c)

Whatever follows from the consequent and from the antecedent follows from the antecedent itself [1, p. 289].

Corollary (2d)

Whatever follows from the consequent with some addition follows from the antecedent with the same addition [1, p. 289]

Sequent rendering

$$\frac{\Rightarrow \phi \to \psi \qquad \Rightarrow (\phi \land \psi) \to \gamma}{\Rightarrow \phi \to \gamma}$$

and

$$\frac{\Rightarrow \phi \to \psi \qquad \Gamma, \psi \Rightarrow \gamma}{\Gamma, \phi \Rightarrow \gamma}$$

Burley's rules for propositional consequences (3)

Rule (3)

In every good nonsyllogistic consequence the [contradictory] opposite of the antecedent follows from the contradictory opposite of the consequent [1, p. 294].

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Given the ambiguity of antecedent and consequent, there are two ways we can represent this as a sequent rule:

$$\frac{\Rightarrow \phi \to \psi}{\Rightarrow \neg\psi \to \neg\phi}$$

or

$$\frac{\phi \Rightarrow \psi}{\neg\psi \Rightarrow \neg\phi}$$

(or even either of these with sets rather than individual propositions).

More corollaries (1)

Corollary (4a)

The contradictory of a conjunctive proposition is equipollent to a disjunctive proposition that has parts that contradict the parts of the conjunctive proposition [1, p. 296].

Corollary (4b)

The contradictory of a disjunctive proposition is equipollent to a conjunctive proposition made up of the contradictories of the parts of the disjunctive proposition [1, p. 297].

Corollary (4c)

The contradictory of a conditional proposition is equivalent to a proposition that signifies that the opposite of its consequent stands together with its antecedent [1, p. 297].

More corollaries (2)

Corollary (4d)

There are two causes of reduplicative propositions; for the contradictory of a reduplicative proposition can be true either because the consequent does not follow from the antecedent or because the antecedent is not the cause of the consequent [1, p. 297].

Corollary (4e)

Although one or the other of a pair of contradictories is affirmed truly of anything whatever considered absolutely, it need not be the case that one or the other of a pair of contradictories is said truly of anything whatever considered under some mode or other [1, p. 297].

Rules for negation (1)

Rule (5)

The negation of any and every [logical] inferior follows from the negation of its [logical] superior [1, p. 298].

Rule (7)

A consequence from a distributed superior to its inferior taken with distribution and without distribution holds good, but a consequence from an inferior to its superior with distribution does not hold good [1, p. 300].

Corollaries for negation (1)

Corollary

Whenever a consequence containing terms taken without distribution is good, it is good the other way around with those terms taken with distribution [1, p. 300].

Corollary

Whenever a consequent follows from an antecedent, the distribution of the antecedent follows from the distribution of the consequent [1, p. 300].

Scope rules for quantifiers and negation

Rule (6)

Negation governs what follows it and not what precedes it [1, p. 298].

Corollary (6a)

A consequence from an inferior to its superior with negation put before does not hold good [1, p. 298].

Corollary (6b)

A consequence from an inferior to its superior using partial or indefinite propositions with negation put after it is good [1, p. 299].

Corollary

A consequence from an inferior to its superior with a word implying negation preceding and affecting both the superior and the inferior does not hold good [1, p. 299].

And much more more!

- Rules based on supposition
- Rules for quantificational and modal reasoning
- Rules for complex predicates