Some Reflections on the History of Connexive Logic

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Three interrelated questions

1. What makes a logic(al system) "connexive"?
2. What historical systems have been called connexive by modern authors?
3. On what grounds?

What makes a logic(al system) "connexive"? (1)

Modernly: A system is called *connexive* if it contains, validates, or proves Aristotle's thesis (syntactic/proof-theoretic view):

"Aristotle's thesis":

$$\neg(\varphi \to \neg\varphi) \tag{1}$$

"Secondary Aristotle's thesis":

$$\neg(\neg\varphi \to \varphi) \tag{2}$$

(Note: $\to$ is a *generic* arrow).

"No formula provably implies or is implied by its own negation" (Wansing 2020).

What makes a logic(al system) "connexive"? (2)

Modernly: *OR* a system is called *connexive* if it contains, validates, or proves Boethius's thesis (syntactic/proof-theoretic view):

"Boethius's thesis":

$$(\varphi \to \psi) \to \neg(\varphi \to \neg\psi) \tag{3}$$

(Note: $\to$ is again a *generic* arrow).

What makes a logic(al system) "connexive"? (3)

A historical alternative:

$$\varphi \rightarrow \psi$$

is true if ψ is somehow "contained" in φ (conceptual containment view).

The relation between the two approaches

Conceptual containment is somehow *prior*, but *leads to* the acceptance of Aristotle's thesis.

Where does "Aristotle's thesis" come from?

Book II, chapter 4 of the *Prior Analytics*:

> *But it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example, that it is impossible that B should necessarily be great if A is white and that B should necessarily be great if A is not white. For whenever if this, A, is white it is necessary that that, B, should be great, and if B is great that C should not be white, then it is necessary if A is white that C should not be white. And whenever it is necessary, if one of two things is, that the other should be, it is necessary, if the latter is not, that the former should not be. If then B is not great A cannot be white. But if, if A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great. But this is impossible. For if B is not great, A will necessarily not be white. If then if this is not white B must be great, it results that if B is not great, it is great, just as if it were proved through three terms.*

“Aristotle’s thesis” vs. Aristotle’s thesis (1)

Two things to note:

1. Not about relationships between *statements* (in the way that $(\varphi \to \psi) \to \neg(\varphi \to \psi)$ is), but about relationships between *terms*. (There is also an explicit reference to *modality*, i.e., this isn’t about accidental relations between terms but necessary ones.)
2. There are two parts to this passage, one, a statement of the thesis, the other a justification of it.

“Aristotle’s thesis” vs. Aristotle’s thesis (2)

The statement of the thesis:

> *it is impossible that B should necessarily be great if A is white and that B should necessarily be great if A is not white.*

If we fudge reference to terms a bit and turn this into something in first-order modal logic, what we have is:

$$\Box\neg(\Box(W\mathrm{a} \to G\mathrm{b}) \wedge (\Box(\neg W\mathrm{a} \to G\mathrm{b}))) \tag{4}$$

With a bit of propositional rearranging, we can get

$$\Box(\Box(W\mathrm{a} \to G\mathrm{b}) \to \neg(\Box(\neg W\mathrm{a} \to G\mathrm{b}))) \tag{5}$$

This is quite a bit different than the statement we gave above.

What justifies the thesis? (1)

The justification is based on the following two claims:

$$\Box(\Box(W\mathrm{a} \to G\mathrm{b}) \wedge (G\mathrm{b} \to \neg W\mathrm{c})) \to \Box(W\mathrm{a} \to W\mathrm{c})) \quad (6)$$

(For whenever if this, A, is white it is necessary that that, B, should be great, and if B is great that C should not be white, then it is necessary if A is white that C should not be white.)

$$\Box((\varphi \to \psi) \to (\neg\psi \to \neg\varphi)) \quad (7)$$

(And whenever it is necessary, if one of two things is, that the other should be, it is necessary, if the latter is not, that the former should not be.)

What justifies the thesis? (2)

Now suppose Aristotle's thesis is *not* true, and we could have both

$$\Box(Wa \to Gb) \tag{8}$$

$$\Box(\neg Wa \to Gb) \tag{9}$$

The argument for Aristotle's thesis then runs as follows:

$$\neg Gb \to \neg Wa \text{ (from 8 and 7)} \tag{10}$$

(If then B is not great A cannot be white.) But from 9, it follows that:

$$\neg Gb \to Gb \tag{11}$$

(But if, if A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great... For if B is not great, A will necessarily not be white. If then if this is not white B must be great, it results that if B is not great, it is great, just as if it were proved through three terms.)

which, per Aristotle, "is impossible."

An alternative justification (1)

McCall, drawing on MacColl: an alternative justification Aristotle's thesis:

Interpret class relationships via implications as follows: Take a random individual from the relevant domain of discourse, and let a represent the claim that this individual belongs to some class A. "All A is B" is then $a \rightarrow b$, and the perfect syllogism Barbara becomes:

$$((a \rightarrow b) \land (b \rightarrow c)) \rightarrow (a \rightarrow c) \tag{12}$$

Exclusions are handled by negating the consequence ("No A is B" is the same as "All A is not B", i.e., $a \rightarrow \neg b$), and overlaps and intersections are handled by the negations of this.

An alternative justification (2)

Now, consider the syllogism Felapton, "If no A is C, and all A is B, then some B is not C", that is:

$$((a \to \neg c) \wedge (a \to b)) \to \neg(b \to c) \tag{13}$$

As McCall notes, this only follows from Barbara "if $\neg(a \to \neg c)$ ("some A is C") follows from $a \to c$ ("All A is C")" [McCall 1967, p. 349]—this is a sort of statement of existential import.

Thus, there is an intimate relationship between Aristotle's thesis and existential import; unsurprising given what we noted above about the fact that 1 holds for any φ that is not contradictory. If some class term is not contradictory then by the "Principle of Plenitude" there must be something inhabiting that class.

Consequences of adopting Aristotle's thesis

What about the formulation as it is generally found in the literature, whether 1 or 2? Both of these are true in classical logic *when φ is contingent*. For the only context in which a formula either implies its negation or is implied by it is when it is either a contradiction or a tautology. Recognizing this makes both 1 and 2 quite sensible indeed: so long as we avoid pathological cases, 1 and 2 describe how the vast majority of sentences work.

Accepting this thesis, however, as a general validity means that we must give up certain classical logical principles, such as *ex impossibile quodlibet* (EIQ). Thus when we look for historical systems that either are connexive or have been described as connexive, we can *also* look for systems where EIQ is admitted; for these, on pain of contradiction, *cannot* be connexive (cf. Johnston 2022).

Boethius's thesis

The "existential import" statement that McCall identified gets us another route to connexivity, via "Boethius's thesis":

$$(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$$

Secondary Aristotle's thesis (2) is a consequence of Boethius's thesis, so a system that admits this thesis will also admit (a form of) Aristotle's thesis, and hence be connexive.

We can ask the same sorts of questions about this as we did about Aristotle's thesis: Where does "Boethius's thesis" come from, and why would people (whether Boethius or others) think it is a reasonable thesis to adopt (that is, in what kinds of contexts is this a true principle)? Lenzen (2020, 2022) has discussed this in detail; we don't here.

The Stoics

So much for the proof-theoretic account: What about the "conceptual containment" account?

This approach is commonly attributed to the Stoics—especially Chrysippus. Chrysippus's view as attributed to him by Sextus Empiricus:

> *And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.*

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However, there is little evidence of their influence (direct or indirect) on medieval developments—my focus.

Historical logics modernly called “connexive”

Throughout the 12th century, discussions of Aristotle’s theses and Boethius’s theses (albeit not explicitly named) are rife.

This is, in part, because the 12th century is a key period in the development of the concept of logical consequence, a working out of the theory of “what follows from what”.

Abelard is, of course, the big name in 12th-century connexive logics (cf. work by Martin and Binini).

Abelard

According to McCall 2012 (pp. 417–418), all of these theses can be found in Abelard:

- Aristotle's first thesis: $\neg(\neg p \to p)$ (sometimes also $\neg(p \to \neg p)$
- Aristotle's second thesis: $\neg((p \to q) \wedge (\neg p \to q))$
- Boethius's thesis: $(p \to q) \to \neg(p \to \neg q)$.
- Abelard's first principle: $\neg((p \to q) \wedge (p \to \neg q))$
- Another Abelardian thesis: $\neg(p \to \neg p)$

Contra Abelard

But other anonymous treatises also discuss connexive principles, or Aristotle's and Boethius's theses, or accounts of consequence that involves the containment of the consequent in the antecedent.

The crux of the dispute between the 12th-century connexivists and the non-connexivists was the paradoxes of material implication: Can something (anything?) follow from what is false (*ex falsa quodlibet*) or impossible (*ex impossibili quodlibet*)? Does a necessity follow from anything, even from things that it has no conceptual connection with?

Authors such as Alberic of Paris and the schools of Robert of Meldun and Adam of Balsham, argued against Abelardian/connexive principles, in favor of EIQ, etc. [Lenzen, 2021].

Adam of Balsham

A prominent voice *in favor* of these rules was Adam of Balsham (Adam de Petit-Pont, Adam Parvipontani). Adam of Balsham's "classical" consequence relation

> *displaced the elaborate theory of hypothetical syllogisms that Boethius proposed, and Peter, after a cursory classification of hypotheticals in LS 1, ignores the whole issue: although his brief discussion of the fallacy of the consequent in LS 7 uses hypothetical propositions as examples, they are never called 'hypotheticals' nor are they linked to the classification of LS 1. Perhaps as a result, his notion of consequentia is underdeveloped in LS 7 and elsewhere, in contrast to theorizing about that topic later in the thirteenth century" [Copenhaver, p. 12–13].*

Curiously, given how it was ultimately his view that won out for centuries to come, the *Ars Disserendi* is not widely studied, and not even translated into English.

The thirteenth century

Other authors began to separate out the distinction between valid conditionals and true consequences. This distinction became increasingly important over the 13th and 14th centuries.

When we look at the thirteenth century, we're faced with a curious question: If Aristotle's principle is a foundational part of his theory, as clearly Boethius took it to be, then why is not pervasive throughout all of medieval logic? In particular, looking at the 13th century and later, the problems that exercises the connexivists of the 12th century seem to have fallen away.

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Why?

The two traditions

The 13th century can be roughly divided into two traditions (Lagerlund & Uckelman 2016):

1. the commentary tradition (based on commentaries on Aristotle)
2. the textbook tradition.

In the former tradition, unsurprisingly, we find discussion of Aristotle's theses; in the latter, not so much.

The commentary tradition: Kilwardby (1)

In the middle of the century, Kilwardby in his commentary on the *Prior Analytics* criticises the thesis, by giving where both p and $\neg p$ appear to imply the same proposition q:

> *If you are seated, God exists.*
> *If you are not seated, God exists [McCall 2012, p. 418].*

(That is, Kilwardby's example is has a necessary consequent—precisely the sort that we said we were excluding above!)

The commentary tradition: Kilwardby (2)

Kilwardby uses this counterexample as motivation to distinguish between two types of implication: essential or natural, and accidental; Aristotle's thesis only applies to the former, not the latter, where it makes sense that we require there to be some sort of essential or natural connection between the parts of the implication. Accidental implications, such as the one above, can be seen as a degenerate form of implication.

This distinction between *consequentiae essentialis* or *naturalis* and *consequentiae accidentalis* is also found in the works of Matthew of Orléans, and a distinction between *naturalis*/*innaturalis* in other 13th-century authors.

The commentary tradition: Kilwardby (3)

But Kilwardby also considers another pair:

> *If you are seated, then either you are seated or you are not seated.*
> *If you are not seated, then either you are seated or you are not seated [McCall, 2012, p. 418].*

These are *consequentia naturalis*, and there *is* a connection between the antecedent and consequent; so the only way that Kilwardby can admit these pairs of implications is if he rejects that Aristotle intended this to fall under the scope of his thesis.

The textbook tradition

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The big four

- William of Sherwood
- Peter of Spain
- Roger Bacon
- Lambert of Auxerre

In the big four, the focus is on conditional statements rather than on consequences. (But even Spruyt's (2018) survey of 13th-century theories of consequences doesn't mention "connexive" or "connexivity".)

The big four

- William of Sherwood: The distinction does not appear in the *Introduction to Logic*
- Peter of Spain: Also not found in *Summaries of Logic*.
- Roger Bacon: The only division of consequences in *The Art and Science of Logic* is this:
- Lambert of Auxerre: The same distinction shows up in the *Summa Lamberti*.

The big four

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- Peter of Spain: Also not found in *Summaries of Logic*.
- Roger Bacon: The only division of consequences in *The Art and Science of Logic* is this:

  > *Note that there are two sorts of consequences: the one inverse, the other direct. A consequence is direct when the opposite of the predicate applies to the opposite of the subject, just as the predicate applies to the subject, as in "As justice is useful, so also injustice is useless". A consequence is inverse when the opposite of the subject applies to the opposite of the predicate as the predicate applies to the subject, as in "If a man is an animal, a non-animal is a non-man". In contradictories, therefore, only inverse consequences arise; in relative and privative opposites, only direct ones. But in contraries, for the most part, direct consequences arise, though in a few cases inverse ones also arise, as Aristotle says, and as an example of which he proposes the following: "If vigor is health, then sickness is debility" [Bacon, ¶181].*

- Lambert of Auxerre: The same distinction shows up in the *Summa Lamberti*.

Lambert on direct vs. inverse consequences

[1321]But someone will object: "No man exists, therefore Caesar does not exist" does indeed seem to follow, for "Caesar exists, therefore a man exists" follows. Therefore by an inverse consequence (which certainly holds with respect to contradictories) it seems that "No man exists, therefore Caesar does not exist" certainly does follow...

[1324] Since mention has been made [1321] of an inverse consequence, in order to know what it is that is called by that name, one should note that according to Aristotle in Book Two of the Topics there are two kinds of consequences, namely, a direct consequence and an inverse consequence. A consequence is direct when the opposite of the consequent follows from the opposite of the antecedent, as in "If it is a strength, it is a virtue; therefore if it is a weakness, it is a vice." A consequence is inverse when the opposite of the antecedent follows from the opposite of the consequent, e.g., "If a man exists, an animal exists; therefore if an animal does not exist, a man does not exist".

Conditionals: William of Sherwood Peter of Spain

What do these texts have to say about conditionals? Are truth/validity conditions of conditionals give in terms of conceptual containment or necessary truth preservation?

Both William and Peter have almost no discussion of hypothetical sentences at all. As Copenhaver, et al., note, this is striking because Peter (and William too) certainly had access to the same historical sources that other, earlier, authors had, including Boethius.

Conditionals: Lambert of Auxerre and Roger Bacon

> *[100] A conditional [proposition] is one whose parts are joined by the connective 'if, as in "If it is a man, it is an animal." It is called conditional by Boethius in his book On Hypothetical Syllogisms because it indicates that something is the case, on the condition that something else is the case. It requires for its truth that the antecedent cannot be true without the consequent [also being true]; but for its falsity it requires that the antecedent could be true without the consequent [also being true].*

Bacon gives the same truth conditions, adding that true consequences are necessary and false ones are impossible [¶161]; and in the construction of the consequence, Bacon cites Boethius *On Hypothetical Syllogisms*.

And yet. . .

Nevertheless, conceptual containment isn't entirely absent in the 13th century and onwards. In treatises that are *neither* commentaries on Aristotle *nor* introductory textbooks we find interesting hints:

- In the anonymous *Obligationes Parisienses*
- In Henry of Ghent (c1217–1293)
- In the *Syncategoremata* treatises and treatises on consequences

Obligationes Parisienses

The early 13th-century *Obligationes Parisiensis* comments that there are "different kinds of inferential relations" [Yrjönsuuri, p. 323]. The author separates *casualis* inferences from necessary ones; the former do not involve a *consequentia naturalis*. According to Yrjönsuuri, the author "seems to mean that no matter how necessary the inferential relation is, it is not grounded in anything like the conceptual inclusion of the consequent in the antecedent, as the standard requirement for natural consequences was at the time often formulated. It is valid, however, since 1 [Socrates is an animal] cannot be true without 2 [Socrates is a man] also being true" [Yrjönsuuri, p. 324].

Thus we begin to see divisions of consequences that accept many different types of consequences as valid, as opposed to the rejection of anything that is not connexive.

Henry of Ghent (1)

> *Henry [denies] that from something impossible anything could follow and that something necessary could follow from anything. As Henry of Ghent puts it, the only way in which something of the antecedent can relate to anything as its cause is if it has some kind of relationship with it, and it is because there is nothing that can have a relationship with everything that therefore there is nothing that can be antecedent to anything or be consequent to anything... he concludes that you can say that from something impossible something can follow, but only if the impossible at issue indicates something that has an intrinsic relationship with the consequent... For the same reason, something true cannot follow from anything whatsoever. The principle that the antecedent cannot be true without the consequent only means, Henry explains, that a conditional is true when the truth conditionalized in the antecedent posits the truth of the consequent [Spruyt, pp. 336–337].*

This has shades of conceptual containment.

Treatises on syncategoremata and consequences (1)

The textbooks in which we found very little to do with connexive logic form an *introductory* basis: But they are not the only teaching-treatises we have. The high point of the 13th and 14th century was the development and establishment of new genres of logic, going beyond the traditional Aristotelian confines:

1. Treatises on syncategoremata
2. Treatises on insolubles and sophisms
3. Treatises on obligations
4. Treatises on consequences

Treatises on syncategoremata and consequences (2)

Treatises on syncategorematic terms usually have an entire sections on the syncategoreme *si* 'if', and how it is used to construct hypothetical/conditional sentences and what the truth conditions of these sentences are.

Conditional sentences are distinguished from consequences, which are not hypothetical in nature but which assert their premises, and connect them to their conclusion with a sign like *ergo* 'therefore'.

Distinguishing between conditionals and consequences means that many more sophisticated distinctions can be made, that were not available to the 12th-century authors.

William of Sherwood's treatise on syncategorematic terms

Now it must be asked, in a conditional, which composition is the one with which truth or falsity occurs? Some say that [truth or falsity] occurs in connection with the composition belonging to 'if' in respect of the verb 'follows' supplied in thought. But on the contrary, an utterance is a sign of a concept; therefore an utterance [is a sign] of a principal concept. Therefore the utterance 'if Socrates is running, he is moving,' is a proposition as far as the principal concept is concerned; therefore truth or falsity will occur in its principal concept; therefore not only in respect of something supplied in thought.

Moreover, Boethius says that when a conditional is negated the negation has to be referred to the verb belonging to the consequent but by including the condition under itself [but Kretzmann can't identify where; fn. 14: Bocheński, "according to the law, which is not expressly formulated:. . . Not if p, then q, if and only if: p and not-q."] [Kretzmann, pp. 118–119].

Classifications of conditionals and consequences

- *essentialis* or *naturalis* vs. *accidentalis* or *innaturalis* (for consequences)
- *formalis* vs. *materialis* (for consequences)
- *simplex* vs. *ut nunc* (for both conditionals and consequences!)

The last distinction isn't the same as "validity as containment" vs. "validity as simple truth-preservation" [Hanke, p. 332]; it's a distinction between "necessary truth" vs. "material truth" (i.e., strict vs. material implication).

The 14th century

A brief look at the 14th century

- Increasing popularity of treatises on consequences
- Continuation of the commentary tradition
- Developments of new ideas.

The commentary tradition in the 14th century

- Radulphus Brito, *Questions on the Prior Analytics*
- John Buridan, *Questions on the Prior Analytics*

While these authors do not go so far as to entirely adopt connexivity, they make use of the distinction between formal and material impossibilities, and formal and material consequences, to restrict the usage of EIQ:

> *[Buridan observes that] if a proposition is formally impossible, then anything you like follows from it formally [Johnston 2022, p. 261].*

But this is not necessarily the case with material consequences.

A new approach to consequences: Marsilius of Inghen (1)

Marsilius of Inghen is a particularly fascinating account. He's at the end of the 14th century and represents the culmination of many different developments. Contra earlier authors such as Buridan, Albert of Saxony, Pseudo-Scotus, etc.,

> *for Marsilius a consequentia is not a propositio, but rather an entailment relation[. This is] clear from his account of the illatio signified by the nota illationis [Ciola 2018, p. 274].*

A new approach to consequences: Marsilius of Inghen (2)

This note functions in two ways:

- On the one hand, it is the grammatical connection between the antecedent and the consequent.
- On the other hand, it signifies the "necessary relation of following of one sentence from another" [Ciola 2018, p. 274].

A new approach to consequences: Marsilius of Inghen (3)

This means that consequences are *always* necessary—*consequentia simplex* rather than *consequentia ut nunc*, though Ciola notes that Marsilius doesn't use that language. This is because for Marsilius, a *consquentia ut nunc* isn't actually a consequence at all [Ciola 2018, p. 286]. And from this we get:

> *Obviously, since the ex falso holds ut nunc, in Marsilius's theory, orationes having a merely false (but not impossible) antecedent are not consequentiae—unless they hold materially for relations of meaning among the terms [Ciola 2018, p. 287].*

Two questions

1. The Kilwardbyan distinction *does* show up in treatises on syncategoremata and consequences: Why not in the textbooks?

2. Why did Adam win over Abelard?

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1. The Kilwardbyan distinction *does* show up in treatises on syncategoremata and consequences: Why not in the textbooks?

   One possible answer: The treatises on syncategoremata and consequences are more advanced. Instead of restricting attention to only one type of inferential relation, without a distinction between conditionals and consequences, as their logical sophistication developed over the 13th and 14th centuries, authors recognized the utility of having many different kinds of conditionals and consequences, suited for solving different types of problems and sophisms—just as we start with material conditionals for intro students and then teach them more complex accounts.

2. Why did Adam win over Abelard?